LECTURE 8

EXTENSION AND INTERPOLATION PROBLEMS IN SOBOLEV SPACES

RECALL THE DEFINITION &

SIMPLEST PROPERTIES

OF

SOBOLEV SPACES.

$L^{m,p}(\mathbb{R}^n)$ CONSISTS OF ALL

$f \in L^p_{loc}(\mathbb{R}^n)$

whose $m^{\underline{th}}$ order distribution deriv's.

$\partial^\alpha f \quad (|\alpha| = m)$

belong to $L^p(\mathbb{R}^n)$.

The $L^{m,p}$ SEMINORM

is given by

$$\|f\| = \sum_{|\alpha|=m} \|\partial^\alpha f\|_{L^p(\mathbb{R}^n)} .$$

NOTE: $\|P\| = 0$ for polys P of degree $< m$.

$W^{m,p}(\mathbb{R}^n)$ CONSISTS OF ALL

$f \in L^p(\mathbb{R}^n)$

whose distribution deriv's

<u>UP TO</u> order m

belong to $L^p(\mathbb{R}^n)$.

The $W^{m,p}$ NORM is given by

$$\|f\| = \sum_{|\alpha| \le m} \|\partial^\alpha f\|_{L^p(\mathbb{R}^n)}.$$

RECALL THE SOBOLEV THM.

If $p > n$, then

$$L^{m,p}(\mathbb{R}^n) \subset C^{m-1}_{loc}(\mathbb{R}^n)$$

and

$$W^{m,p}(\mathbb{R}^n) \subset C^{m-1}(\mathbb{R}^n).$$

If $p > \frac{n}{m}$, then

$$L^{m,p}(\mathbb{R}^n) \subset C^0_{loc}(\mathbb{R}^n)$$

and

$$W^{m,p}(\mathbb{R}^n) \subset C^0(\mathbb{R}^n).$$

NOTE:

STRICTLY SPEAKING, FUNCTIONS IN $L^{m,p}$ OR $W^{m,p}$ ARE DEFINED ONLY UP TO A SET OF MEASURE ZERO.

For $p > n/m$,

Sobolev $\Longrightarrow$

May redefine any $F \in L^{m,p}(\mathbb{R}^n)$

on a set of measure 0

to make it continuous.

SET

$X = L^{m,p}(\mathbb{R}^n)$ or $W^{m,p}(\mathbb{R}^n)$.

LET $\| \; \|_X$ DENOTE THE

CORRESPONDING

SEMINORM

or

NORM.

GIVEN $E \subset \mathbb{R}^n$, LET

$$X(E) = \{ F|_E : F \in X \},$$

WITH (SEMINORM or NORM)

$$\| f \|_{X(E)} = \text{INF} \left\{ \|F\|_X : \begin{array}{l} F \in X, \\ F = f \text{ on } E \end{array} \right\}.$$

For $p > n/m$, this makes sense, thanks to Sobolev thm.

For $p \leq n/m$,

$X(E)$ NEEDN'T MAKE SENSE,

because $F \in X$ needn't be

continuous.

(Try $E = \{\text{SINGLE PT}\}$.)

NOTE:

IF E IS FINITE, THEN

$$X(E) = \{\text{all } f : E \to \mathbb{R}\},$$

but WE STILL WANT TO UNDERSTAND

$$\|f\|_{X(E)}.$$

We write $[X(E)]^*$

to denote the space

of linear functionals on $X(E)$.

(Use this when E is finite.)

WE WILL ALWAYS

ASSUME HERE THAT

$p > n$,

BUT OUR QUESTIONS

MAKE SENSE FOR

$p > n/m$

WHITNEY'S PROBLEMS

FOR

SOBOLEV SPACES

DECIDE WHETHER A GIVEN

$f: E \to \mathbb{R}$

belongs to $X(E)$.

Given $f \in X(E)$,

Compute the order of magnitude of

$\|f\|_{X(E)}$.

DOES THERE EXIST A

BOUNDED LINEAR MAP

$T: X(E) \to X$

SUCH THAT

$Tf = f$ on E

for all $f \in X(E)$?

Suppose $E \subset \mathbb{R}^n$ is

<u>FINITE</u>.

$\boxed{\begin{array}{l}\text{Given } f: E \to \mathbb{R}, \\ \text{Compute a fn. } F \in X \\ \text{such that} \\ \quad F = f \text{ on } E \\ \quad\text{and} \\ \quad \|F\|_X \leq C \|f\|_{X(E)}.\end{array}}$

PREVIOUS WORK

ON

WHITNEY'S PROBLEM

for

SOBOLEV SPACES

GREAT CREDIT GOES TO:

PAVEL SHVARTSMAN,

WHO UNDERSTOOD

WHITNEY'S PROBLEMS

FOR

$L^{1,p}(\mathbb{R}^n)$, $p > n$.

ARIE ISRAEL,

WHO UNDERSTOOD

THE CASE OF

$L^{2,p}(\mathbb{R}^2)$ $(p>2)$

AND INTRODUCED THE

KEY IDEA of

"KEYSTONE CUBES" (*)

(*) (SEE BELOW)

KEVIN LULI

and

PAVEL SHVARTSMAN,

who UNDERSTOOD

$L^{m,p}(\mathbb{R}^1)$.

TODAY'S LECTURE STATES THE RESULTS OF JOINT WORK OF

ARIE ISRAEL,
KEVIN LULI,
&
cf.

The next lecture
will sketch the proofs
of some of the results
stated today.

NOTATION:

CONSTANTS DENOTED BY

$c, C, C', \ldots$

ALWAYS DEPEND ONLY

on

m, n, p.

(RECALL: $X = L^{m,p}(\mathbb{R}^n)$ or $W^{m,p}(\mathbb{R}^n)$)

THE SIMPLEST VERSION

OF OUR RESULTS

IS AS FOLLOWS.

RECALL :

WE ALWAYS ASSUME THAT

$p > n$.

Thm 1: Given $E \subset \mathbb{R}^n$, there exists a linear map

$T : X(E) \to X$

such that for all $f \in X(E)$ we have

$$\boxed{\begin{array}{l} Tf = f \text{ on } E \\ \text{and} \\ \|Tf\|_X \le C \|f\|_{X(E)}. \end{array}}$$

THM 2 (COMPUTATION of the NORM):

Suppose $E \subset \mathbb{R}^n$ is FINITE,

$N = \#(E)$.

Then there exist linear functionals

$$\xi_1, \xi_2, \cdots, \xi_L : X(E) \to \mathbb{R}$$

SUCH THAT :

PROPERTIES OF $\xi_1, \dots, \xi_L$

$$\boxed{L \leq CN}$$

For any $f \in X(E)$, the number

$$|||f||| = \left(\sum_{\ell=1}^{L} |\xi_\ell(f)|^p \right)^{1/p}$$

satisfies

$$c|||f||| \leq \|f\|_{X(E)} \leq C|||f|||$$

For finite $E \subset \mathbb{R}^n$,

the linear map

in Thm 1,

and the functionals

in Thm 2

have **additional structure**

that will help us to

compute efficiently.

We prepare to introduce the notion of

ASSISTED

BOUNDED

DEPTH

("ABD")

MOTIVATION

GIVEN N REAL NUMBERS

$x_1, \ldots, x_N$,

LET'S COMPUTE THEIR

VARIANCE, σ^2.

TWO STANDARD FORMULAS

$$\boxed{\sigma^2 = \frac{1}{2N^2} \sum_{i,j=1}^{N} (x_i - x_j)^2}$$

(NEEDS $\sim N^2$ COMPUTER OPS.)

$$\sigma^2 = \frac{1}{N} \sum_{i=1}^{N} (x_i - \bar{x})^2$$

where

$$\bar{x} = \frac{1}{N} \sum_{i=1}^{N} x_i$$

(NEEDS $\sim N$ COMPUTER OPS.)

Moral:

It pays to

precompute.

LET'S SEE WHAT WE WILL PRECOMPUTE for Whitney's Problems in Sobolev Spaces.

LET $E \subset \mathbb{R}^n$ be FINITE, $\#(E) = N$.

A LINEAR FUNCTIONAL

$$\omega : X(E) \to \mathbb{R}$$

can be expressed as

$$\boxed{\omega(f) = \sum_{x \in E} \mu(x) \cdot f(x)}$$

with coefficients $\mu(x)$ independent of f.

WE DEFINE THE DEPTH OF ω,

DENOTED BY $dp(\omega)$,

TO BE THE NUMBER OF

NONZERO COEFFICIENTS $\mu(x)$.

Given $f \in X(E)$, it takes

$\sim dp(\omega)$ COMPUTER OPS.

to CALCULATE $\omega(f)$.

A FINITE SET Ω OF LINEAR FUNCTIONALS on $X(E)$ WILL BE CALLED A SET OF ASSISTS FOR $X(E)$ IF IT SATISFIES

$$\boxed{\sum_{\omega \in \Omega} dp(\omega) \leq CN.}$$

Let Ω be a set of assists for $X(E)$.

$\boxed{\text{Given } f \in X(E) \text{, the WORK required to evaluate } \underline{\underline{\text{ALL}}} \text{ the functionals } \omega \in \Omega \text{ on } f \text{ is comparable to the work required MERELY TO READ THE DATA } f.}$

THE ASSISTS $\omega \in \Omega$

will play the rôle of

the MEAN $\overline{X}$

in the computation of the

variance of N given

real numbers.

WE NOW GIVE THE

DEFINITION OF

ASSISTED BOUNDED

DEPTH.

Let $E \subset \mathbb{R}^n$, $\#(E) = N < \infty$.

Let $\Omega \subset [X(E)]^*$ be a set of assists.

A linear functional $\xi \in [X(E)]^*$ has

Ω-assisted bounded depth

if it can be written as follows ...

$$\boxed{\xi(f) = \sum_{x \in E} \mu(x) f(x) + \sum_{\omega \in \Omega} \lambda(\omega) \cdot \omega(f)}$$

where

$\mu(x)$ $(x \in E)$ and $\lambda(\omega)$ $(\omega \in \Omega)$ are INDEPENDENT of f

and

AT MOST C OF THE COEFFICIENTS $\mu(x)$, $\lambda(\omega)$ are NONZERO.

SUPPOSE WE ARE GIVEN :

- $E \subset \mathbb{R}^n$ finite
- $\Omega \subset [X(E)]^*$ a SET OF ASSISTS
- $\xi \in [X(E)]^*$ of Ω-assisted BOUNDED DEPTH
- $f \in X(E)$.

$$\boxed{\text{ONCE WE HAVE PRECOMPUTED } \omega(f) \text{ for all the } \omega \in \Omega, \text{ WE CAN CALCULATE } \xi(f) \text{ USING ONLY } C \text{ COMPUTER OPS.}}$$

NEXT WE DEFINE

OPERATORS OF

Ω- ASSISTED

BOUNDED DEPTH.

Let $E \subset \mathbb{R}^n$ (finite),

and let $\Omega \subset [X(E)]^*$

be a set of assists for $X(E)$.

A linear map $T : X(E) \to X$

has $\boxed{\Omega\text{-assisted bdd. depth}}$

if it can be expressed

as follows.

$$Tf(x) = \sum_{y \in E} \mu(x,y) f(y) + \sum_{\omega \in \Omega} \lambda(x,\omega) \cdot \omega(f)$$

for all $f \in X(E)$ and $x \in \mathbb{R}^n$,

where

$\mu(x,y)$ and $\lambda(x,\omega)$ are INDEPENDENT of f,

and

For each fixed $x \in \mathbb{R}^n$, at most C of the coefficients $\mu(x,y)$, $\lambda(x,\omega)$ are NONZERO.

SUPPOSE WE ARE GIVEN

- $E \subset \mathbb{R}^n$ FINITE
- $\Omega \subset [X(E)]^*$, A SET OF ASSISTS
- $T : X(E) \to X$,
  A LINEAR MAP OF Ω-ASSISTED BDD DEPTH.
- $f \in X(E)$.

ONCE WE HAVE PRECOMPUTED

$\omega(f)$ (all $\omega \in \Omega$),

we can calculate $Tf(x)$

for any given QUERY PT. x

USING AT MOST C COMPUTER OPS.,

ASSUMING SOMEONE TELLS US

THE COEFFICIENTS

$\mu(x, y)$ and $\lambda(x, \omega)$.

NOTE :

$\Omega = \phi$ (EMPTY SET)

$\Longrightarrow$

Ω-ASSISTED BDD. DEPTH

REDUCES TO OUR EARLIER

NOTION of

BOUNDED DEPTH.

WE CAN NOW STATE

TODAY'S MAIN RESULTS

IN A MORE

PRECISE FORM.

THM 3: LET $E \subset \mathbb{R}^n$, FINITE.

Then there exist a SET OF ASSISTS

$\Omega \subset [X(E)]^*$

and a LINEAR MAP

$T : X(E) \to X$

OF Ω-ASSISTED BOUNDED DEPTH,

such that for all $f \in X(E)$

we have

$$\boxed{\begin{array}{l} Tf = f \text{ on } E \\ \text{and} \\ \|Tf\|_X \le C \|f\|_{X(E)} \end{array}}$$

<u>Thm 4</u>: Let $E \subset \mathbb{R}^n$, $\#(E) = N < \infty$.

Then there exist

$$\Omega \subset [X(E)]^*$$

and

$$\xi_1, \dots, \xi_L \in [X(E)]^*$$

with the following properties:

- Ω is a SET OF ASSISTS
- Each ξ_ℓ has Ω-ASSISTED BDD. DEPTH
- $L \le CN$

$\boxed{\text{Moreover, for each } f \in X(E), \text{ the number } |||f||| = \left(\sum_{\ell=1}^{L} |\xi_\ell(f)|^p \right)^{1/p} \text{ satisfies } c|||f||| \le \|f\|_{X(E)} \le C|||f|||.}$

QUESTION:

Can we replace

ASSISTED BDD DEPTH

by

BDD. DEPTH

in

Thms 3 & 4?

ANSWER:

FOR THM 3 ON

LINEAR EXTENSION OPERATORS

the ANSWER IS

NO!

(WE HAVE A COUNTEREXAMPLE.)

For Thm 4

(on the computation of $\|f\|_{X(E)}$),

the ANSWER IS

MAYBE.

WE DON'T KNOW.

THERE'S AN INTERESTING

POSSIBLE CONNECTION

TO ...

SPARSIFICATION

BATSON-SPIELMAN-SRIVASTAVA PROVED THE FOLLOWING RESULT.

Thm of BSS :

Let $\xi_1, \dots, \xi_N$ be linear functionals on $\mathbb{R}^D$.

Then there exist coefficients $\lambda_1, \dots, \lambda_N \geq 0$ such that

At most CD of the λ_n are NONZERO,

↑ ABSOLUTE CONSTANT

yet

$$\sum_n \lambda_n \cdot (\xi_n(v))^2 \leq \sum_n (\xi_n(v))^2 \leq 2 \sum_n \lambda_n (\xi_n(v))^2$$

for all $v \in \mathbb{R}^D$.

In particular, we can calculate the variance of $x_1, \dots, x_N \in \mathbb{R}$ up to a factor of 2, using $\mathcal{O}(N)$ COMPUTER OP's WITHOUT PRECOMPUTING.

REMARKS:

- The results of BATSON-SPIELMAN-SRIVASTAVA are MORE PRECISE than the version stated here.

- The $O(N)$ computation of a variance without precomputing follows from the existence of EXPANDER GRAPHS.

Thanks to

BO'AZ KLARTAG

&

ASSAF NAOR

for discussions of

Sparsification.

QUESTION, POSSIBLY RELEVANT TO WHITNEY'S PROBLEMS IN SOBOLEV SPACES:

$\boxed{\text{IS THERE A VERSION OF THE BSS THM, with } \sum_n |\xi_n(v)|^2 \text{ replaced by } \sum_n |\xi_n(v)|^p \text{ ?}}$

NO ONE KNOWS

SEE ALSO THE WORK OF

PAVEL SHVARTSMAN

on

SPARSIFICATION

for $L^{m,p}(\mathbb{R}^n)$

when $m = 1, 2$.

We now turn from SPARSIFICATION to discuss further Thms 3 & 4 on LINEAR EXTENSION OPS. & COMPUTATION of the NORM.

Thm 5: Using at most

$CN \log N$ computer ops

and at most CN storage,

we can compute a

set of ASSISTS Ω for $X(E)$

and a set of functionals

$\xi_1, \dots, \xi_L$ satisfying the

Conditions of Thm 4.

Recall the ASSERTIONS OF THM 4

$\boxed{\Omega \text{ is a set of assists}}$

$\boxed{\text{Each } \xi_\ell \text{ has } \Omega\text{-assisted bdd. depth.}}$

$\boxed{L \leq C N}$

The quantity

$$|||f||| = \left(\sum_{\ell=1}^{L} |\xi_\ell(f)|^p \right)^{1/p}$$

differs from

$$\| f \|_{X(E)}$$

by at most a factor C

(any $f \in X(E)$).

Corollary of Thm 5

Given $E \subset \mathbb{R}^n$ with $\#(E) = N < \infty$,

WE can do $O(N \log N)$ work,

AFTER WHICH, GIVEN ANY $f \in X(E)$

WE CAN COMPUTE $\|f\|_{X(E)}$

UP TO A FACTOR C,

USING $O(N)$ COMPUTER OPS.

(IT TAKES N OP'S JUST TO READ f.)

<u>THM 6</u>: UNDER THE ASSUMPTIONS OF THM. 3, WE CAN <u>COMPUTE</u> A SET OF ASSISTS $\Omega \subset [X(E)]^*$ AND A LINEAR MAP

$$T: X(E) \rightarrow X$$

SATISFYING THE CONCLUSIONS OF THM. 3.

THE COMPUTATION IS <u>EFFICIENT</u>.

MORE PRECISELY:

$$\boxed{\text{WE CAN COMPUTE } \Omega \text{ USING } \le CN\log N \text{ COMPUTER OP's, and } \le CN \text{ STORAGE.}}$$

FURTHERMORE,

AFTER ONE-TIME WORK $\le CN\log N$

USING STORAGE $\le CN$,

WE CAN ANSWER QUERIES

A QUERY CONSISTS OF A POINT $x \in \mathbb{R}^n$.

The RESPONSE to a query x is a formula

$$Tf(x) = \sum_{y \in E} \mu(x,y) f(y) + \sum_{\omega \in \Omega} \lambda(x,\omega)\, \omega(f),$$

where we omit terms with $\mu(x,y)$ or $\lambda(x,\omega) = 0$.

$\boxed{\text{The WORK TO ANSWER A QUERY IS AT MOST } C \log N.}$

Our linear map T satisfies

$Tf = f$ on E

and

$\|Tf\|_X \leq C \|f\|_{X(E)}$

for all $f \in X(E)$.

END OF THM. 6

Corollary:

Let $E \subset \mathbb{R}^n$, $\#(E) = N < \infty$.

Given $f: E \to \mathbb{R}$,

We can perform

$\leq C N \log N$ work, using

$\leq C N$ storage,

after which we can

answer Queries.

A QUERY CONSISTS of A POINT $x \in \mathbb{R}^n$,

and the RESPONSE to a query x is $F(x)$, for a function $F \in X$ such that

$F = f$ on E

and

$\|F\|_X \leq C \|f\|_{X(E)}$.

Here, F is determined by f.

(It doesn't depend on our QUERIES.)

$$\boxed{\begin{array}{l}\text{Proof of the Cor:} \\ \text{Take } F = Tf, \\ \text{and apply Thm 6 to} \\ \text{compute } T.\end{array}}$$

The above results bring our UNDERSTANDING of WHITNEY's PROBLEMS for $W^{m,p}(\mathbb{R}^n)$, $L^{m,p}(\mathbb{R}^n)$ close to what was achieved for $C^m(\mathbb{R}^n)$,

$\boxed{\text{PROVIDED } p > n}$.

WHAT ABOUT

$p \leq n$?

Of course we require

$p > \frac{n}{m}$, $p \geq 1$.

The Next Lecture

will sketch the Proofs

of Some of today's Thms.

THANK YOU!